Mid-Semester Exam — ISI Bangalore — 16th Sept 2013

Elementary Differential Geometry — 100 Points

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously. I am allotting **4 points** for this.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{R}$ = real numbers.

---

1. [24 points] Prove that the following maps $F\colon U \to V$ are C^∞ diffeomorphisms.
(a) $U = \mathbb{R} = V$, $F(x) = x + x^3$.
(b) $U = \mathbb{R}^2 = V$, $F(x, y) = (x, y + x^3)$.
(c) $U = \mathbb{R}^2 = V$, $F(x, y) = (x + p(x) - y, y - p(x))$ where $p(x) \in C^\infty(\mathbb{R})$.

2. [24 points] For any $b \neq 0$, the parametrized plane curve $\alpha(t) = (e^{bt}\cos t, e^{bt}\sin t)$ is called a logarithmic spiral.
   (i) Compute the curvature $\kappa(t)$.
   (ii) For any fixed b, verify that the angle between $\alpha(t)$ and the tangent vector $\dot{\alpha}(t)$ is constant.
   (iii) For $b = 1$, find a unit-speed reparametrization of α.

3. [16 points] Let $\alpha(t)$ be a C^∞ curve in $\mathbb{R}^n$.
   (i) With $n = 4$, if $\alpha'(0) = (1, 0, -1, 1)$ and $\alpha''(0) = (2, 1, 0, -1)$, compute the corresponding unit tangent vector $T(0)$, principal normal $N(0)$ and curvature $\kappa(0)$.
   (ii) With $n = 3$, if $\alpha'(0) = (1, 0, -1)$, $\alpha''(0) = (2, 1, 1)$ and $\alpha'''(0) = (3, -1, 2)$ compute $T(0)$, $N(0)$, $B(0)$, $\kappa(0)$, $\tau(0)$.

4. [16 points]
   (i) Give an example of a regular parametrized curve in $\mathbb{R}^3$ that has constant nonzero curvature and constant nonzero torsion everywhere.
   (ii) If $\alpha(s)$ is a unit-speed curve in $\mathbb{R}^3$ with nonzero curvature such that $\ddot{N}(s)$ is parallel to $N(s)$ for every s, then prove that the curvature $\kappa(s)$ and torsion $\tau(s)$ are constant.

5. [16 points] Suppose $\sigma(x, y) = (\sigma_1(x, y), \sigma_2(x, y), \sigma_3(x, y))$ is a C^∞ map from U to $\mathbb{R}^3$ where U is an open neighborhood of $(0, 0) \in \mathbb{R}^2$. Assume that $(\sigma_{1x}\sigma_{2y})(0, 0) \neq (\sigma_{1y}\sigma_{2x})(0, 0)$.
   (i) Apply the inverse function theorem to prove that there is a $\mathcal{C}^\infty$ function $h(s, t)$ over some open neighborhood $U' \subset U$ of $(0, 0)$ such that $\sigma_3(x, y) = h(\sigma_1(x, y), \sigma_2(x, y))$ over U'.
   (ii) Assume that $\sigma(0, 0) = (0, 0, 0)$. Fix (a_1, a_2, a_3) in $\mathbb{R}^3$ not parallel to $(\sigma_x \times \sigma_y)(0, 0)$, and let H denote the plane $a_1T_1 + a_2T_2 + a_3T_3 = 0$. Prove that there is a neighborhood $U' \subset U$ of $(0, 0)$ such that $H \cap \sigma(U')$ is a regular parametrized curve.